$$(\vec{u}\times\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k}$$

$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$|\vec{u}\times\vec{v}| = \sqrt{4+16+9} = \sqrt{29}$$

$$\therefore \quad \text{S.D.} = \left| (\vec{r_2} - \vec{r_1}) \frac{(\vec{u}\times\vec{v})}{|\vec{u}\times\vec{v}|} \right|$$

$$= \frac{|(\hat{j}-4\hat{k}).(2\hat{i}-4\hat{j}-3\hat{k})|}{\sqrt{29}} = \frac{8}{\sqrt{29}} \text{ unit.}$$

13 LINEAR PROGRAMMING

IMPORTANT FORMULAE

- **Solution of a Linear Programming Problems :** A set of values of the variables $x_1, x_2, x_3, ..., x_n$ satisfying the constraints of a linear programming problem is called a solution of the Linear Programming Problem.
- **Objective Functions :** If $c_1, c_2, c_3, ..., c_n$ are constants and $x_1, x_2, x_3, ..., x_n$ are variables then the linear function $z = c_1x_1 + c_2x_2 + ... + c_nx_n$ which we have to maximise or minimised is called objective function.
- **Write in mathematical form of the L.P.P. Questions :**
  (i) Make a table to given datas.
  (ii) Noted the objective variable.
  (iii) Prepare objective function.
  (iv) Noted all doing work in problem.
- **Solution of L.P.P. by the graphical method :**
  (i) Convert into equations to given inequations.
  (ii) The lines are drawn by joining these given points on both axes.
  (iii) Find the values of the objective function Z at each of the extreme points.

➡ Long Answer Type Questions

Q. 1. A gold smith manufactures necklace and bracelets the total no. of necklaces and bracelets that he can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. It is assumed that he can work for a maximum of 164 hrs. day. Further the profit on a bracelet is ₹ 300 and the profit on a necklace is 100. Find how many of each should be produced daily to maximize the total profit ?

Solution :

The given L.P.P. can be stated mathematically as :

$$\text{Max } Z = 100x_1 + 300x_2$$

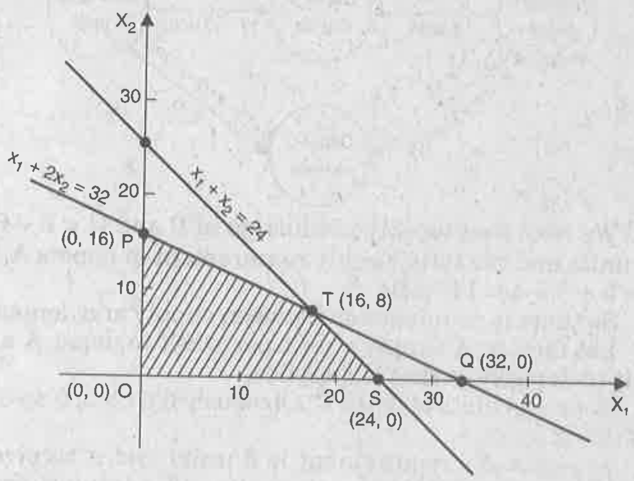
s.t.

$$x_1 + 2x_2 \le 32$$
$$x_1 + x_2 \le 24$$

and non-negative restrictions,

$$x_1 \ge 0, x_2 \ge 0$$

Proceeding stepwise, the permissible region is the shaded region OSTPO which is the set of all points which simultaneously satisfy all the constaints and the non-negative restrictions. Solving simultaneously, the equations of the corresponding intersecting lines, we get the co-ordinates of the vertices of the shaded region as P (0, 16), O (0, 0), S (24, 0), T (16, 8).



Now, the values of the objective function at these corner points are as given in the table below :

Point (x_1, x_2)	Value of $Z = 100x_1 + 300x_2$
P (0, 16)	$100 \times 0 + 300 \times 16 = 4800$ (Max.)
O (0, 0)	$100 \times 0 + 300 \times 0 = 0$
S (24, 0)	$100 \times 24 + 300 \times 0 = 2400$
T (16, 8)	$100 \times 16 + 300 \times 80 = 4000$

Clearly Z is max. at P (0, 16). Hence $x_1 = 0$ and $x_2 = 16$ is the optimal solution of the given problem and the optimal value of Z is 4800. *i.e.*, the goldsmith earns maximum profit when he makes 16 bracelets and 0 necklace a day and that his maximum profit in this case is ₹ 4800.

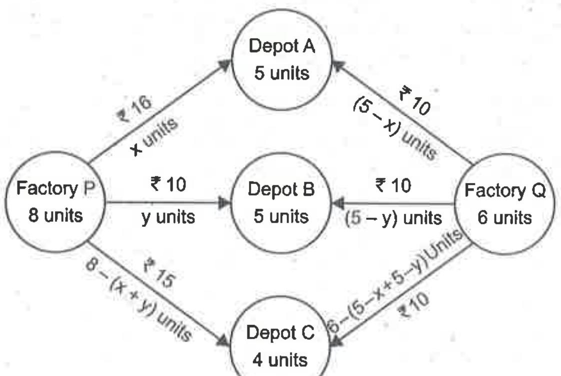
Q. 2. A company has two factories located at P and Q and has three depots situated at A, B and C, the weekly requirements of the depots of A, B, C is respectively 5, 5 and 4 units while the production capacity of the factories at P and Q are 8 and 6 units respectively. The cost of transportation per unit is given below :

From \ To	A	B	C
	Cost (in ₹)		
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum ?

Solution :

The given data can be represented diagrammatically as :



We note that weekly production of P and Q = 8 + 6 = 14 units and the total weekly requirement at depots A, B, C = 5 + 5 + 4 = 14 units.

So there is no mismatch between supply and demand.

Let factory A supply x unit per week to depot A and unit to depot B so that it supplies.

$(8-x-y)$ units to depot C. Obviously $0 \le x \le 5$, $0 \le y \le 5$, $0 \le (8-x-y) \le 4$.

As depot A's requirement is 5 units and it receive x units from factory P it must receive $(5-x)$ units from factory Q.

Similarly depot B receives $(5-y)$ units from factory Q and depot C receives $4-(8-x-y) = x+y-4$ units from factory Q. As a cross check, quantity supplied from factory Q to depot A, B, C = $(5-x)+(5-y)+(x+y-4) = 6$

= capacity of Factory Q

Thus, the total transportation cost in ₹ is

$$= 16x + 10y + 15(8-x-y) + 10(5-x) + 12(5-y) + 10(x+y-4)$$
$$= x - 7y + 190$$

Hence the given problem can be formulated as an L.P.P. as

$$Z = x - 7y + 190$$

s.t. $\quad x + y \le 4$

$x + y \ge 8$

$x \ge 0, x \le 5$

$y \ge 0, y \le 5.$

Q. 3. A house wife wishes to mix together two kind of food X & Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamin B & 8 units of vitamin C. The vitamin contents of one kg of food is given below :

	Vitamin A	***Vitamin B***	***Vitamin C***
Food X :	**1**	**2**	**3**
Food Y :	**2**	**2**	**1**

One kg of food X costs ₹ 6 and one kg of food Y costs ₹ 10. Find the least cost of the mixture which will produce the diet.

(*CBSE*, 2003, *Compartment*, 2009)

Solution :

Let x kg of food X and y kg of food Y are mixed together to make the mixture.

Since one kg of food X contains one unit of vitamin A and one kg of food Y contains 2 units of vitamin A therefore, kg. of food X and kg of food Y will contain $x + 2y$ units of vitamin A. But the mixture should contain atleast 10 units of vitamin A.

So, $\quad x + 2y \ge 10$

Similarly, kg. of food X and kg of food Y will produce $2x + 2y$ units of vitamin B and $3x + y$ units of vitamin C. But the minimum requirements of vitamin B and C are respectively of 12 and 8 units.

$\therefore \quad 2x + 2y \ge 12$ and $3x + y \ge 8$

Since the quantity of food X and food Y cannot be negative

$\therefore \quad x \ge 0, y \ge 0$

It is given that one kg of food X costs ₹ 6 and one kg of food Y costs ₹ 10. So x kg of food X and y kg of food Y will cost ₹ $(6x + 10y)$.

Thus, the given L.P.P. is

Min. $\quad Z = 6x + 10y$

s.t. $\quad x + 2y \ge 10$

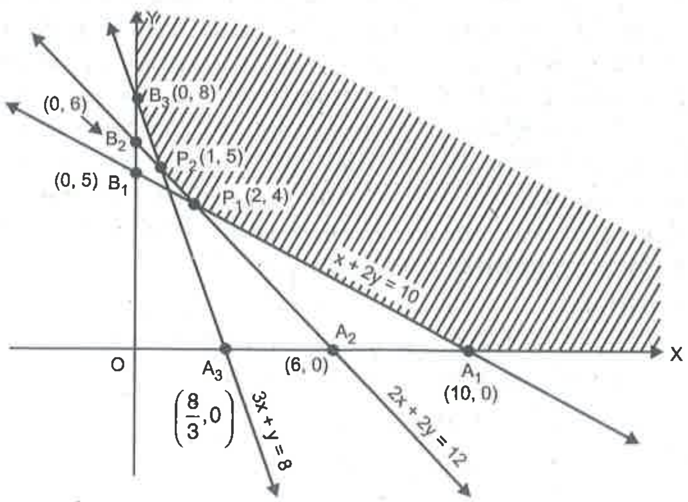
$2x + 2y \ge 12$

$3x + y \ge 8$

and $\quad x \ge 0$ & $y \ge 0$

To solve this L.P.P. we draw lines $x + 2y = 10$, $2x + 2y = 12$ and $3x + y = 8$.

The feasible region of the L.P.P. is shaded in the fig. below :



The co-ordinates of the vertices (corner-points) of shaded feasible region $A_1P_1P_2B_3$ are A_1 (10, 0), P_1 (2, 4), P_2 (1, 5) and B_3 (0, 8). Thses points have been obtained by solving the equations of the corresponding intersecting lines the values of objective function are as follows :

Point (x, y)	*Value of Z = 6x + 10y*
A_1 (10, 0)	Z = 6 × 10 + 10 × 0 = 60
A_2 (2, 4)	Z = 6 × 2 + 10 × 4 = 52
P_2 (1, 5)	Z = 6 × 1 + 10 × 5 = 56
B_3 (0, 8)	Z = 6 × 0 + 10 × 8 = 80

Clearly, Z is minimum at $x = 2$ and the minimum value of Z is 52.

Hence, the least cost of the mixture is ₹ 52.

Q. 4. A dealer wishes to purchase a no. of fans and sewing machines. He has only ₹ 5,760 to invest and has space for atmost 20 items. A fan costs him ₹ 360 and a sewing machine ₹ 240. His expectation is that he can sell a fan at a profit of ₹ 22 and a sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ?

[*CBSE, AI, Compartment (Delhi)*, 2009]

Solution :

Suppose the dealer buys x fans & y sewing machines, since the dealer has space for atmost 20 items. Therefore,

$$x + y \le 20$$

A fan costs ₹ 360 and a sewing machine costs ₹ 240. Therefore, total cost of x fans and y sewing machines is ₹ $(360x + 240y)$. But the dealer has only ₹ 5760 to invest so,

$$360x + 240y \le 5760$$

Since the dealer can sell all the items that he can buy and the profit on a fan is of ₹ 22 and on a sewing machine the profit is of ₹ 18. Therefore, total profit on selling x fans and y sewing machines is of ₹ $(22x + 18y)$.

Let Z denotes the total profit then $Z = 22x + 18y$

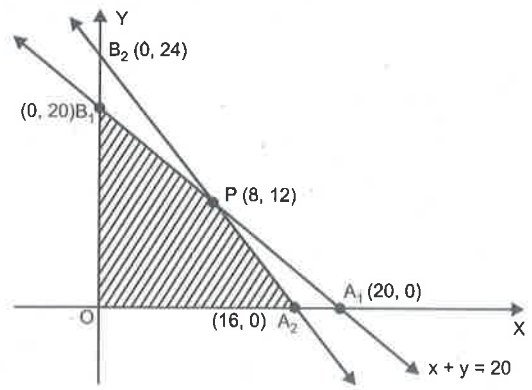
Clearly $x, y \ge 0$

thus the mathematical formulation of the given problem is

Max. $Z = 22x + 18y$

s.t. $x + y \le 20$

$360x + 240y \le 5760$



and $x \ge 0, y \ge 0$

To solve this L.P.P. graphically, we first convert the inequations into the equations and draw the corresponding lines. The feasible region of the L.P.P. is shaded in fig. The Corner points of the feasible region OA_2PB_1, are O (0, 0), A_2 (16, 0) P (8, 12) & B_1 (0, 20).

Points (x, y)	*Value of Z = 22x + 18y*
O (0, 0)	Z = 22 × 0 + 18 × 0 = 0
A_2 (16, 0)	Z = 22 × 16 + 18 × 0 = 352
P (8, 12)	Z = 22 × 8 + 18 × 12 = 392
B (0, 20)	Z = 22 × 0 + 18 × 20 = 360

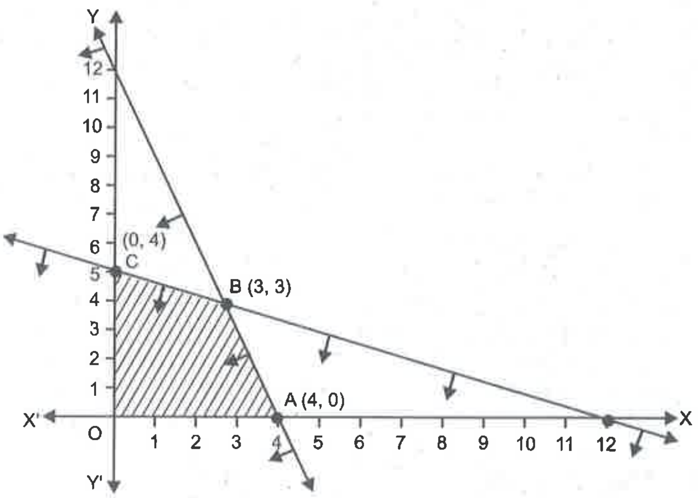
Clearly, Z is Max. at $x = 8$ and $y = 12$.

& also Maximum value of Z is 392.

Q. 5. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts while it takes 3 hour on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 17.50 per package of nuts and ₹ 7.00 per package on bolts. How many packages of each should be produced each day so as to maximize his profit if he operates his machines for atmost 12 hours a day ? (*Compartment, AI*, 2009, *CBSE, Delhi*, 2012)

Solution :

Let x packages of nuts and y packages of bolts must be produced.



Hence Max. $Z = 17.50x + 7y$

$x + 3y \le 12$ (machine A constraints)

$3x + y \le 12$ (machine B constraints)

$x, y \ge 0$ (non-negative restrictions)

Now, using the graph, we get the feasible region is OABC.

Now calculating at each corner point.

Corner Point	*Value of Z*
(0, 0)	0
(4, 0)	70
(3, 3)	73.50 (Max.)
(0, 4)	28

Three packets nuts and three packets of bolts.
Maximum profit 73.50.

Q. 6. A diet is to contain atleast 80 units of vitamin A and 100 units of minerals, two foods F_1 and F_2 are available. Food F_1 cost ₹ 4 per unit and food cost ₹ 6 per unit. A unit of F_1 contains atleast 3 units of vitamin A and 4 units of minerals. A unit of food F_2 contains atleast 6 units of vitamin A and 3 units of minerals. Formulate this is a linear programming problem. Find the minimum cost for diet that consist of mixture of these two foods and also meet the minimal nutritional requirements.

(*CBSE*, *Delhi*, 2009)

Solution :

Let x units of food F_1 and y units of F_2 must be produced.

Then Min. $Z = 4x + 6y$ (cost function)

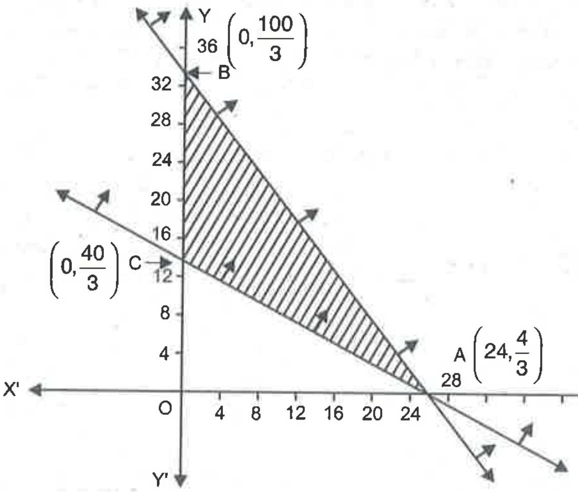
s.t. $3x + 6y \geq 80$ (vitamin A constraints)

$4x + 3y \geq 100$ (vitamin B constraints)

$x, y \geq 0$ (non-negative restrictions)

Now, drawing the graph of these inequations, we get the feasible region is ABC.

Now, calculating Z at each corner point, we have



Corner Point	*Value of Z*
$\left(24, \frac{4}{3}\right)$	104
$\left(0, \frac{100}{3}\right)$	200
$\left(0, \frac{40}{3}\right)$	80 (Min.)

Hence only $\frac{40}{3}$ units of food F_2 must be produced to minimize the cost.

Q. 7. One kind of cake requires 300 gm of flour and 15 gm of fat another kind of cake requires 150 gm flour and 30 gm of fat. Find the maximum cakes which can be made from 7.5 kg of flour and 600 gm of fat, assuming that there is no shortage of the other ingradients used in making the cakes. Make it as an L.P.P. and solve it graphically.

(*CBSE*, *AI*, 2010)

Solution :

Let 1st kind of cake be x and 2nd kind of cake be y.

We requires the flour and fats as follows :

Kind of Cake	*No. of Cakes*	*Flour*	*Fat*
I	x	300 g	15 g
II	y	150 g	30 g
Total		7,500 g	600 g

Then, we have : $Z = x + y$

Requirement of flour $= 300x + 150y$

Total requirement of flour $= 7{,}500$ gm

$\Rightarrow$ $300x + 150y \leq 7{,}500$

$\Rightarrow$ $2x + y \leq 50$

Requirement of fat $= 15x + 30y$

Total requirement of fat $= 600$ gm

$\therefore$ $15x + 30y \leq 600$

$x + 2y \leq 40$ ...(ii)

So given conditions $2x + y \leq 0, x + 2y \leq 40, x, y \geq 0$

Q. 8. Solve the linear programming problem by graphical method.

Max. $z = 3x_1 + 2x_2$

such that $x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$ (*USEB*, 2010)

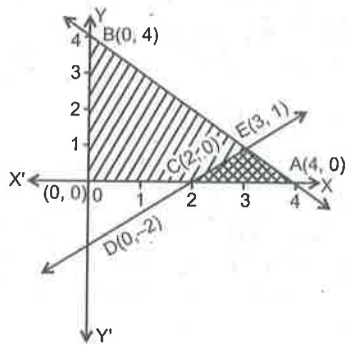
Solution :

Given, Max. $z = 3x_1 + 2x_2$

such that $x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$



(1) $x_1 + x_2 \leq 4$ (graph)

Line $x_1 + x_2 = 4$ passes through the points A (0, 4) and B(4, 0).

$\therefore$ $x = 0$ and $x_2 = 0$ in $x_1 + x_2 \leq 4$,

$\therefore$ $0 \leq 4$ which is true.

$\therefore$ Origin O(0, 0) lie in this field.

(2) Graph of $x_1 - x_2 \leq 2$

Line $x_1 - x_2 = 2$ passess through the points C(2, 0) and D (0, – 2)

Putting $x_1 = 0$ and $x_2 = 0$ in $x_1 - x_2 \leq 2$.

$\therefore 0 - 0 \leq 2$ which is true.

Now calculating Z at each corner point, we get

Corner Points	*Value of Z*
(0, 0)	0 (Min.)
(0, 4)	8
(3, 1)	11 (Max.)
(2, 0)	6

Hence, maximum value of Z is 11 at (3, 1).

Q. 9. A small
chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minute to make a chain. The maximum number of hours available per day is 16. If the profit on a ring is ₹ 300 and that on a chain is ₹ 190, find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it as on L.P.P. and solve it graphically. (*CBSE, Delhi*, 2010)

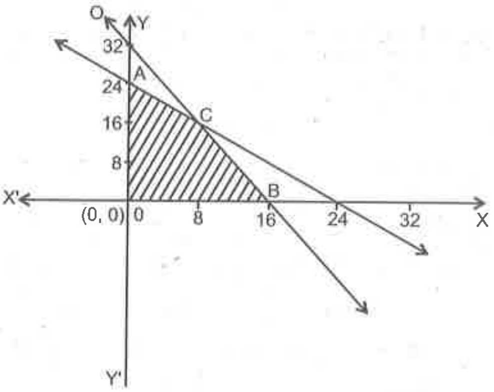
Solution :

Let the no. of rings and chains manufactured be x and y.

Given conditions, $x + y \leq 24$ ...(1)

$60x + 30y \leq 960$ ...(2)

$x, y \geq 0$ ...(3)



Let the profit be

$$P = 300x + 190y$$

From (1), $$\frac{x}{24} + \frac{y}{24} = 1$$

From (2), $$\frac{x}{16} + \frac{y}{32} = 1$$

Solving eqs. (1) and (2), we get

$$x = 8, y = 16$$

Now calculating P at each corner point, we get

Corner Points	*Value of P*
(0, 0)	0
(0, 24)	4,560
(16, 0)	4,800
(8, 16)	5,440

Hence, maximum profit should be earned at (8, 16) is 5,440.

Q. 10. A diet for a sick person must contain atleast 4,000 units of vitamin, 50 units of minerals and 1,400 units of calories. Two foods A and B are available at a cost of ₹ 4 and ₹ 3 per unit respectively, if one unit of A contains 200 units of vitamin, 1 unit of minerals and 40 units of calories, and one unit of food B contains 100 units of vitamin, 2 units of minerals and 40 units of calories. Find what combination of food should be used to have the least cost. (*CBSE*, 2008)

We can make the following table from the given data :

Resources	*Food*		*Requirement*
	A	*B*	
Vitamin	200	100	4,000
Minerals	1	2	50
Calories	40	40	1,400
Cost	4	3	

If x units of food A and y units of food B are mixed then the total cost $C = 4x + 3y$, clearly $x \geq 0$ and $y \geq 0$.

Since there must be atleast 4,000 units of vitamins in the diet,

$$200x + 100y \geq 400$$

i.e. $$2x + y \geq 35$$

Similarly, as there must be atleast 50 units of minerals in the diet, $x + 2y \geq 50$

and as there must be atleast 1,400 calories in the diet

$$40x + 40y \geq 1,400,$$

i.e. $$x + y \geq 35$$

Hence, the given diet problem can be formulated as an L.P.P. as follows :

Minimize $Z = 4x + 3y$

s.t. $2x + y \geq 40$

$x + 2y \geq 50$

$x + y \geq 35$

and $x \geq 0, y \geq 0.$

Q. 11. A factory owner purchases two types of machines, A and B, for his factory. The requirements and limitations for the machines are as follows :

	Area occupied by the machine	*Labour force for each machine*	*Daily output in units*
Machine A	1,000 sq. m.	12 men	60
Machine B	1,200 sq. m.	8 men	40

He has an area of 9,000 sq. m. available and 72 skilled men who can operate the machines. How many machines of each type should he buy to maximize the daily output? (*CBSE, Delhi*, 2008)

Solution :

Let A and B types machines purchase x and y and getting daily output be Z, then

$$Z = 60x + 40y \quad ...(i)$$

Getting maximum area = 9,000 m² ...(ii)

$\therefore$ $1,000x + 1,200y \leq 9,000$

$\Rightarrow$ $5x + 6y \leq 45$

No. of labourers = 72 men

$\therefore$ $12x + 8y \leq 72$

$\Rightarrow$ $3x + 2y \leq 18$ ...(iii)

Now in the form of LPP,

Max. $Z = 60x + 40y$

$5x + 6y \leq 45$

$3x + 2y \leq 18$

$x \geq 0$ and ≥ 0

Now $5x + 6y = 45$

$\Rightarrow$ $$\frac{x}{9} + \frac{2y}{15} = 1$$

Thus the corner points are :

$$O(0, 0), C(6, 0), E\left(\frac{9}{4}, \frac{45}{8}\right) B\left(0, \frac{15}{2}\right)$$

Price of daily output Z = 60x + 40y

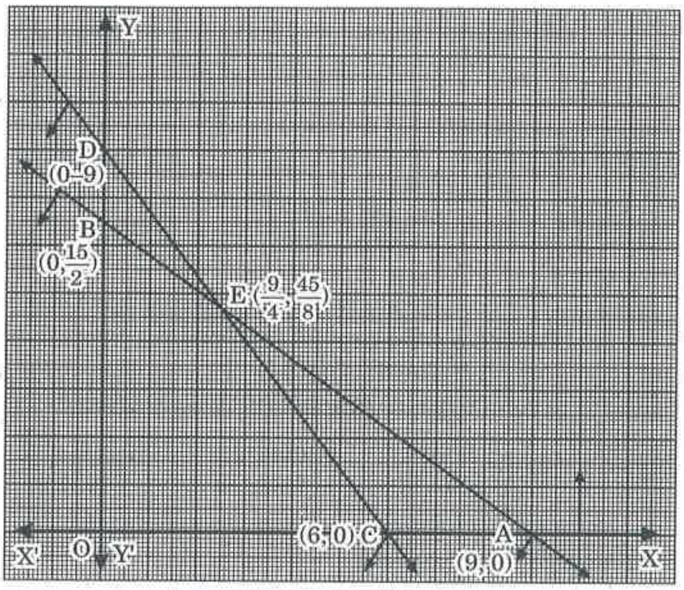
(i) At O (0, 0), Z = 0

(ii) At C(6, 0), Z = 360

(iii) At E $\left(\frac{9}{4}, \frac{45}{8}\right)$ Z = 360

(iv) At B$\left(0, \frac{15}{2}\right)$, Z = 300

Hence A type machine = 6 and B type machine = 0 or A type machine = 2 and B type machine = 6 use is daily output.4



NCERT QUESTIONS

Q. 1. A merchant plans to sell two types of personal computers—a desktop model and a portable model that will cost ₹ 25,000 and ₹ 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the no. of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakh and if his profit on desktop model is ₹ 4,500 and on portable model is ₹ 5,000.

Solution :

Let x unit of desktop and y units of portable must be produced to get maximum profit.

Then Max. Z = 4500x + 5000y (profit function)

s.t. 25,000x + 40,000y ≤ 70,00,000 (cost constraints)

$x + y \le 250$ (demand constraints)

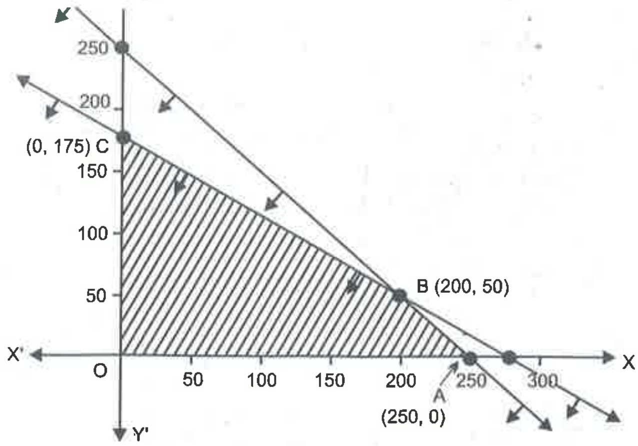
$x, y \ge 0$ (non-negative restrictions)

or Max Z = 4500x + 5000y

s.t. $5x + 8y \le 1400$

$x + y \le 450$

$x, y \ge 0$



Now, drawing the graph we get the feasible region is OABC.

Now calculating the value of at Z each corner point.

Corner Point	*Value of Z*
(0, 0)	0
(250, 0)	11,25,000
(200, 50)	11,50,000 (Max.)
(0, 175)	8,75,000

Hence, 200 units of desktop and 50 units of portable must be purchased to get maximum profit of ₹ 11,50,000.

Q. 2. A factory makes tennis rackets and cricket bat. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftsman time in a day. The factory has availability of not more than 42 hours of machine time and 24 hours of craftsman time.

(i) What no. of rackets and bats must be produced if the factory is to work at full capacity?

(ii) If the profit on a racket and on a bat is ₹ 20 and ₹ 10 respectively, find max. profit.

Solution :

Let the no. of rackets produced = x

and that of bats = y

Then we have Max. Z = 20x + 10y

s.t. $1.5x + 3y \ge 42$ (machine constraints)

$3x + y \le 24$ (craftman constraints)

$x, y \ge 0$ (non-negative restrictions)

or we can write as

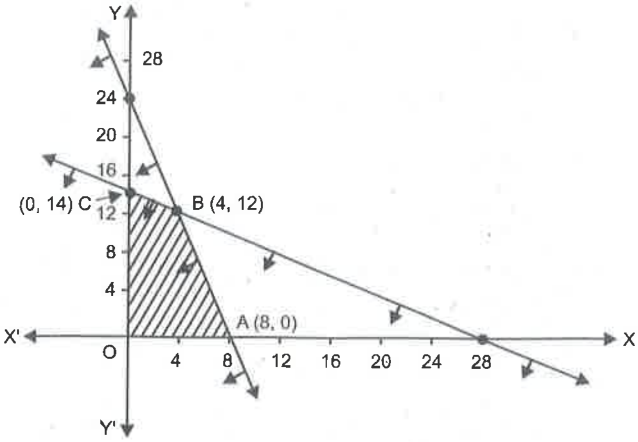
Max. Z = 20x + 10y

s.t. $x + 2y \le 28$

$3x + y \le 24$

$x, y \ge 0$

Now, drawing the graph, we get



Now calculating Z at each corner point of the feasible region OABC, we get

Corner Point	*Value of Z*
(0, 0)	0
(8, 0)	160
(4, 12)	200 (Max.)
(0, 14)	140

4 tennis rackets and 12 cricket bats, maximum profit = ₹ 200.

Q. 3. There are two types of fertilisers F_1 and F_2 consist of 10% nitrogen and 6% phosphoric acid and contains 5% nitrogen and 10% phosphoric. Acid after testing the soil condition a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If F_1 cost ₹ 6 per kg and F_2 cost ₹ 5 per kg, determine how much of each type of fertilizer should be used so that nutrients requirements are met at a minimum cost. What is the minimum cost ?

Solution :

Let x kg of F_1 and y kg of F_2 are mixed.

then Min. $Z = 6x + 5y$ (cost function)

s.t. $\frac{10x}{100} + \frac{5y}{100} \geq 14$ (phosphoric acid constraint)

$\frac{6x}{100} + \frac{10y}{100} \geq 14$ (Phosphoric acid constraint)

$x, y \geq 0$ (non-negative restriction)

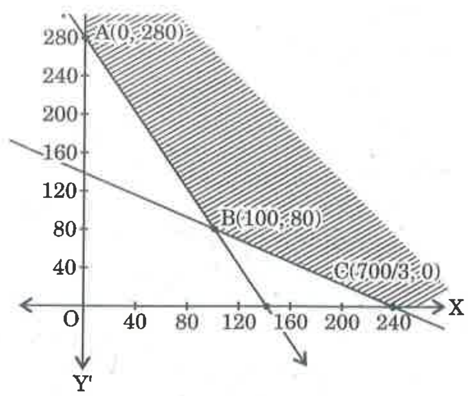
or Min $Z = 6x + 5y$

s.t. $x + 2y \geq 280$

$3x + 5y \geq 700$

$x, y \geq 0$

Now, drawing the graph of the inequation, we get the feasible region is ABC which is not bounded.



Now, calculating Z at each corner point.

Corner Point	*Value of Z*
(0, 280)	1400
(100, 80)	1000 (Min.)
$\left(\frac{700}{3}, 0\right)$	1400

Hence, 100 kg of F_1 and 80 kg of F_2 must be used to minimize to cost upto 1000.

Q. 4. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below :

Food	*Vitamin A*	*Vitamin B*	*Vitamin C*
X	1	2	3
Y	2	2	1

1 kg of food X cost ₹ 16 and 1 kg of food Y cost ₹ 20. Find the least cost of the mixture which will produce require diet.

Solution :

Let x units of food X and y units of food Y must be mixed.

Then Min $Z = 16x + 20$ (cost function)

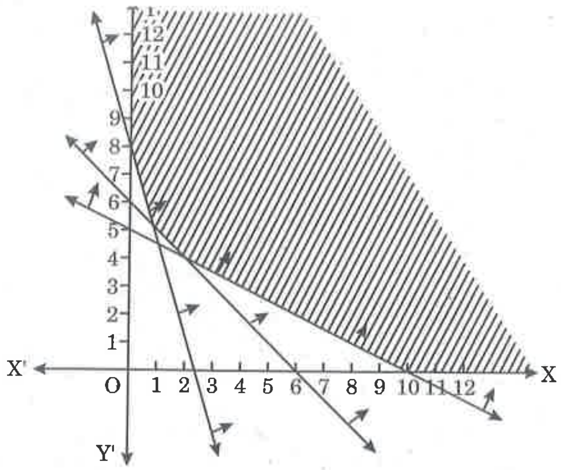
s.t. $x + 2y \geq 0$ (vitamin A constraints)

$2x + 2y \geq 12$ (vitamin B constraints)

$3x + y \geq 8$ (vitamin C constraints)

and $x, y \geq 0$ (non-negativity constraints)

Now drawing the graph of the inequation and finding the feasible region.



The feasible region is unbounded with the corner pairs $(0, 8), (10, 0), (1, 5), \left(\frac{6}{5}, \frac{22}{5}\right)$.

Now, calculating Z at each corner point :

Corner Point	*Value of Z*
(0, 8)	160
(10, 0)	160
(1, 5)	116
$\left(\frac{6}{5}, \frac{22}{5}\right)$	$\frac{536}{5}$ (Min.)

Hence, 1 unit of food x and 53 units of must be taken. Cost = 192.

Q. 5. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 1,000 is made on each executive class ticket and a profit of ₹ 600 is made on each economy class ticket. The airline reserves atleast 20 seats for executive class. However, atleast 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximize the profit for the airline. What is the maximum profit ?

Let x passenger of executive class and y passengers of economy class must be taken.

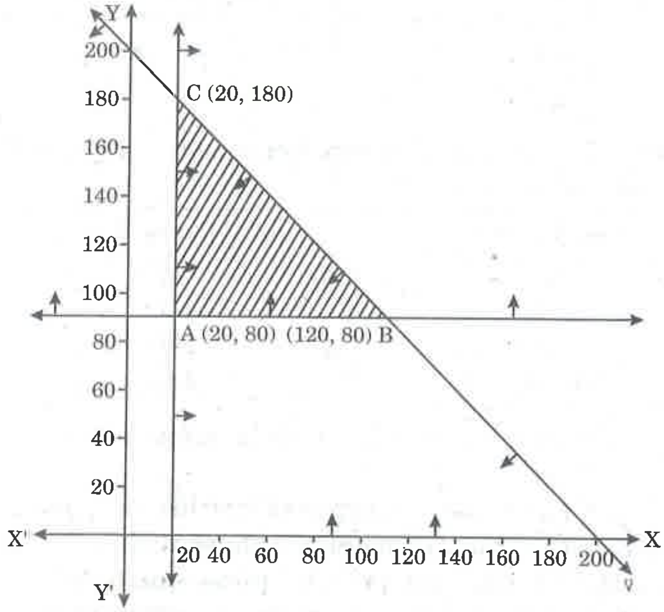
Then Max $Z = 1000x + 600y$ (Profit function)

s.t. $x + y \leq 200$ (capacity constraints)

$x \geq 20$

$y \geq 4x$

Now, drawing the graph of the above we get the feasible region is ABC.



Now calculating the value of Z at each corner point when :

Corner Point	*Value of Z*
(20, 80)	68,000
(120, 80)	1,68,000 (Max.)
(20, 180)	1,28,000
(40, 160)	1,36,000

Hence, 120 tickets of executive class and 80 tickets of economy class must be sold.

Maximum profit = ₹ 1,68,000.

■■

14 PROBABILITY

IMPORTANT FORMULAE

PROBABILITY

- Sample space is the set of all possible functions of random experiments.
- Probability of any event, $P(A) = \frac{n(A)}{n(S)}$.
- For any event A, $0 \leq P(A) \leq 1$ and if A = S, then P (A) = 1.
- Supplement of A is A' or $\bar{A}$ or A^C.

  P(nor A) or = $P(A') = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  or P(A or B) = P(A) + P(B) – P(A and B)

  Also, P (neigher A nor B) = 1 – P(A or B)
- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

- **Mutually Exclusive Events :** Two ... are said to be mutually exclusive if one of them occurs, others can not occur. Thus two or more events are said to be mutually exclusive if no two of them can occur together. Thus events $A_1, A_2, ..., A_n$ are mutually exclusive if and only if

$$A_i \cap A_j = \phi \text{ for } i \neq j$$

- **Exhaustive Events :** For a random experiment, a set of events is said to be exhaustive if one of them must necessarily happen every time the experiment is performed.

  For example :

  when a dice is thrown events {1}, {2}, {3}, {4}, {5}, {6} form an exhaustive set of events.
- **Independent Events :** Two events E and F are said to be independent, if P(E/F) = P(E) and P(F/E) = P(F), provided P (E) $\neq$ 0 P(F) $\neq$ 0 and P (E) $\neq$ 0.

$$P(E \cap F) = P(E) \times P(F)$$

- **Conditional Probability :** Let E and F be two events with a random experiment. The probability of occurrence of E under the condition that F has already occured and P(F) $\neq$ 0 is called the conditional probability. It is represented by $P\left(\frac{E}{F}\right)$.

$$\therefore \quad P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)}$$

when $\quad P(E) \neq 0$

- $$\text{Var}(X) = E(X^2) - [E(X)]^2$$
- A random experiment with exactly two possible results is called a Binominal or Bernoulli trial.

  A Binomial experiment is experiment that can be regarded as a sequence of n trials in which :

  (i) n is finite and is defined before the experiment begins,

  (ii) each trial has only two possible outcomes, usually called success and failure,

  (iii) the result of any trial is independent of the results of all other trials and

  (iv) the probability of success does not change from trial to trial.

Very Short Answer Type Questions

Q. 1. A coin is tossed 400 times and it shows head 200 times. Discuss whether the coin is unbiased or biased.

Solution :

Here $p = \frac{1}{2}, q = \frac{1}{2}$, $n = 400$, we locate that n is large.

$$\text{Mean value} = np = 400\left(\frac{1}{2}\right) = 200$$

and $\quad$ Standard Deviation $\sigma = \sqrt{npq}$

$$= \sqrt{400.\frac{1}{2}.\frac{1}{2}} = 10$$

Mean value = 220 – 200 + 2 (10) = Mean + 2σ

Hence, the coin is biased.

Q. 2. ... out of 10 is busy ? If telephone numbers are randomly selected and called, find the probability that four of them will be busy ?

Solution :

Let event A be telephone dialed and busy, then

$$p = P(A) = \frac{1}{10} \text{ and } q = \frac{9}{10}$$

Thus, the required probability = ${}^nC_r\, p^r\, q^{n-r}$

$$= {}^6C_4\left(\frac{1}{10}\right)^4\left(\frac{9}{10}\right)^2$$

$$= \frac{6.5.4.3}{1.2.3.4}.\frac{1}{10^6}\times 81$$

$$= \frac{15.81}{10^6} = \frac{243}{2.(10)^5} = 0.001215.$$

Q. 3. Two coins (or one coin twice) are tossed. If X denotes the number of tails, find the random variable X for this random experiment.

Solution :

Let sample space of random experiment be S :

$$S = [HH, HT, TH, TT]$$

$$X(HH) = 0, X(HT) = 1, X(TH) = 1, X(TT) = 2.$$

Q. 4. A die is thrown twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 2 has appeared atleast once ?

Solution :

Sample space, S = 36

Let A = sum of the numbers be 7

= {(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)}

and B = number 2 has appeared atleast once

= {(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2), (6, 2)}

and thus A ∩ B = {(2, 5), (5, 2)]

$$\therefore P(A) = \frac{6}{36} = \frac{1}{6}, P(B) = \frac{11}{36}, P(A \cap B) = \frac{2}{36} = \frac{1}{18}$$

We want to find $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{18}}{\frac{1}{6}} = \frac{6}{18} = \frac{1}{3}.$

Q. 5. If A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cup B) = \frac{2}{3}$. Are the events A and B independent ?

Solution :

Total sum of probability,

P (A ∪ B) = P (A) + P (B) – P (A ∩ B)

$$\Rightarrow \quad \frac{2}{3} = \frac{1}{2} + \frac{1}{3} - P(A \cap B) \text{ (Putting the value)}$$

$$\Rightarrow \quad P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{2}{3} = \frac{3+2-4}{6} = \frac{1}{6}$$

Again P (A). P(B) = $\frac{1}{2}.\frac{1}{3} = \frac{1}{6}$ = P (A ∩ B)

Hence, the events A and B are independent.

Short Answer Type Questions

Q. 1. A family has two children. What is the probability that both the children are boys, given that atleast one of them is a boy ? (*CBSE, AI*, 2010)

Solution :

Let denoted girl by G and Boy by B, then

Sample space of experiment will be

S = [BB, BG, GB, GG]

Let event E = both are boys

and event F = atleast one boy

Thus E = [BB] and

F = [BG, GB, BB]

$F \cap F = [BB] \therefore P(E) = \frac{1}{4}, P(F) = \frac{3}{4}$

and $P(E \cap F) = \frac{1}{4}$

$$\therefore \quad P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}}$$

$$\therefore \quad P(E/F) = \frac{1}{3}.$$

Q. 2. A die is thrown twice and the sum of the number appearing observed to be 6. What is the conditional probability that the number 4 has appeared atleast once ? (*JAC*, 2010)

Solution :

Let event

E = sum of event 6

F = 4 has appeared atleast once.

Sample point of event E = [(1, 5), (2, 4), (3, 3), (4, 2), (5,1)]

Sample point of event F = [(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (1, 4), (2, 4), (3, 4), (5, 4), (6, 4)]

Thus $E \cap F$ = [(2, 4), (4, 2)]

No. of elements in sample space = 6 × 6 = 36

O (E) = 5, O (F) = 11 and O $(E \cap F)$ = 2

$$\therefore \quad P(E) = \frac{5}{36}, P(F) = \frac{11}{36} \text{ and } P(E \cap F) = \frac{2}{36}$$

According to question,

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{2/36}{5/36} = \frac{2}{5}.$$

Q. 3. A speaks truth 2 out of 3 times and B speaks truth 4 times out of 5. They agree in assertion that from a bag containing 6 balls of different colours, a red ball has been drawn. Find the probability that the statement is true. (*USEB*, 2010)

Solution :

Condition I. Red ball drawn and A, B both speak true. Its possibilities :

(speak true)

$$= \frac{1}{6}.\frac{2}{3}.\frac{4}{5} = \frac{4}{45}$$

Condition II. Red ball has not drawn but A, B speak lie that ball is red :

P (no red ball), P (A speaks lie), P (B speaks lie)

$$= \frac{5}{6}\left(\frac{1}{3}.\frac{1}{5}\right)\left(\frac{1}{3}.\frac{1}{5}\right)$$

$$= \frac{1}{450}$$

$$\therefore \quad \text{Required probability} = \frac{\frac{4}{45}}{\frac{4}{45}+\frac{1}{450}} = \frac{40}{11}.$$

Q. 4. A man is known to speak truth 3 times out of 4. He throws a die and reports that it is a six. Find the probability that it is actually a six. (*USEB*, 2009)

Solution :

Let E_1 = 6 comes, E_2 = 6 does not come and

A = man says that it is 6

$$P(E_1) = \frac{1}{6} \text{ and } P(E_2) = \frac{5}{6}$$

Now $P(A/E_1)$ = Probability getting 6 on a dice

$= \frac{3}{4}$

and $P(A/E_2)$ = Probability do not getting 6 on a dice

$= 1 - \frac{3}{4} = \frac{1}{4}$

Bays' Theorem

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{3/24}{8/24} = \frac{3}{8}$$

Q. 5. On a multiple choice question with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing ? (*CBSE, AI* 2009 & *Delhi*, 2010)

Solution :

Let the number of correct answers be P :

$$P = \frac{1}{3}$$

and $$q = \frac{2}{3}$$

$$P(X \geq 4) = P(x = 4) + P(x = 5)$$

$$= {}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) + {}^5C_5\left(\frac{1}{3}\right)^0\left(\frac{1}{3}\right)^5$$

$$= 5\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)^5$$

$$= \frac{5\times 2}{243}+\frac{1}{243}=\frac{11}{2}.$$

Q. 6. Prove that :

$$P(A/B) = \frac{P(B/A)\,P(A)}{P(B/A).\,P(A)+P(B/\bar{A})P(\bar{A})}$$

(*USEB*, 2010)

Solution :

We know that

$$P(A\cap B) = P(B/A).P(A) \quad ...(i)$$

and

$$= P(B/A).P(A) \quad ...(ii)$$

$\because$ A and $\bar{A}$ such that

$$A\cap \bar{A} = \phi$$

and

$$A\cup \bar{A} = S$$

$\therefore$

$$P(B) = P(B/\bar{A}).P(A) + P(B/\bar{A}).P(\bar{A})$$

Putting this in equation (ii),

$P(A\cap B) = P(A/B)[P(B/A)P(A) + P(B/\bar{A})\,P(\bar{A})]$ ...(iii)

From equations (i) and (iii),

$P(B/A)\,P(A) = P(A/B)[P(B/A)\,P(A) + P(B/\bar{A})\,P(\bar{A})]$

$$P(A/B) = \frac{P(B/A)P(A)}{P(B/A)P(A)+P(B/\bar{A})P(\bar{A})}.$$ **Proved.**

Q. 7. Probability that a boy passes an examination is $\frac{3}{5}$ and that for a girl it is $\frac{2}{5}$. Find the probability that atleast one of them passes this examination. (*USEB*, 2010)

Solution :

Probability a boy passes an exam $= \frac{2}{5} = P(A)$ (Let)

Probability a girl passes an exam $= \frac{2}{5} = P(B)$ (Let)

Probability both pass an exam

$$P(A\cap B) = P(A).P(B)$$

$$= \frac{3}{5}\times\frac{2}{5}=\frac{6}{25}$$

$\because$ Events are mutualy independent.

$\therefore$ Probability that atleast one of them passes exam

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= \frac{3}{5}+\frac{2}{5}-\frac{6}{25}=1-\frac{6}{25}=\frac{19}{25}$$

Q. 8. A bag contains 5 white, 7 red and 4 black balls. On drawing three balls at random, find the probability of being all white. (*USEB*, 2010)

Solution :

Total No. of balls in a bag = 5 + 7 + 4 = 16

No. of white balls = 5

No. of balls drawing 3 white balls from 16 balls

$$= \frac{16!}{3!13!}$$

No. of balls drawing 3 white balls from 5 white balls

$$= \frac{5!}{3!2!}$$

Probability of being all white balls

$$= \frac{5!}{3!2!}\Big/\frac{16!}{3!13!}$$

$$= \frac{5!\times 13!}{16!\times 2!}$$

$$= \frac{5\times 4\times 3}{16\times 15\times 14}=\frac{1}{56}$$

Q. 9. A bag A contains 3 white and 2 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that the ball was drawn from the bag B. (*CBSE Delhi*, 2004, 05)

Solution :

Let E_1 = Event of A bag selected

E_2 = Event of B bag selected

E = Event of drawing red ball

$\because$ Red ball drawn from bag B, so find $\left(\frac{E_2}{E}\right)$.

$\because$ Events of both bags selected are equilikely, then

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E_1/E) = \frac{2}{5} \text{ and } P(E_2/E) = \frac{5}{9}$$

$$\therefore \quad P\left(\frac{E_2}{E}\right) = \frac{P(E_2)P\left(\frac{E_2}{E}\right)}{P(E_1)P(E_1/E)+P(E_2)P(E_2/E_1)}$$

or

$$P\left(\frac{E_2}{E}\right) = \frac{\frac{1}{2}\times\frac{5}{9}}{\frac{1}{2}\times\frac{2}{5}+\frac{1}{2}\times\frac{5}{9}}$$

$$= \frac{\frac{5}{18}}{\frac{1}{5}+\frac{5}{18}} = \frac{\frac{5}{18}}{\frac{18+25}{90}} = \frac{5}{18}\times\frac{90}{43}=\frac{25}{43}$$

Q. 10. How many times must a man toss a fair coin, so that the probability of having atleast one head is more than 80%. (*CBSE, Delhi*, 2012)

Solution :

$\because$ For a fair coin,

$$P(A) = \frac{1}{2} \text{ and } P(B) = \frac{1}{2}$$

Here A and B represented by head and tail.

Let the coin be tossed n times.

Required Probability = 1 – P (all tails)

$$= 1-\frac{1}{2^n}$$

It is greater than 80% or $\frac{80}{100}=\frac{4}{5}$.

We know that, total probability = 1

$$\therefore \quad 1-\frac{1}{2^n} > \frac{4}{5}$$

$$\Rightarrow \quad 1-\frac{4}{5} > \frac{1}{2^n}$$

$$\Rightarrow \quad \frac{1}{5} > \frac{1}{2^n}$$

$$\Rightarrow \quad 2^n > 5$$

Least value of n is 3.

Q. 11. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of red cards. *(CBSE, Outside Delhi, 2012)*

Solution :

Let X represents the no. of red cards in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Now, $P(X = 0) = $ P (no red card)

$= $ P (two balck cards)

$$= \frac{^{26}C_2}{^{52}C_2} = \frac{25}{102}$$

$P(X = 1) = $ P(one red card and 1 black card)

$$= \frac{^{26}C_1 \times {}^{26}C_1}{^{52}C_2} = \frac{25}{102}$$

and $P(X = 2) = $ P(two red cards)

$$= \frac{^{26}C_2}{^{52}C_2} = \frac{25}{102}$$

Thus the probability distribution of X is :

X	0	1	2
P(X)	$\frac{25}{102}$	$\frac{52}{102}$	$\frac{25}{102}$

Now, the mean of X = E(X)

$$= \sum_{i=1}^{n} x_i P(x_i)$$

$$= 0 \times \frac{25}{102} + 1 \times \frac{52}{102} + 2\frac{25}{102} = 1$$

Also, $$E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i)$$

$$= 0^2 \times \frac{25}{102} + 1^2 \times \frac{52}{102} + 2^2 \times \frac{25}{102}$$

$$= \frac{152}{102}$$

$$\therefore \quad Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{152}{102} - (1)^2$$

$$= \frac{50}{102} = \frac{25}{52}$$

➠ Long Answer Type Questions

Q. 1. Find the mean, variance and standard deviation of the number of heads in two tosses of a coin.

Solution :

Sample space S = [HH, HT, TH, TT]

Randon varable X = no. of heads, so, it takes 0, 1, 2.

$$P(0) = P(TT) = \frac{1}{4}$$

$$P(1) = P(HT \text{ or } TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$P(2) = P(HH) = \frac{1}{4}$$

We prepare following table :

x_i	0	1	2	
p_i	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	
$p_i x_i$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\Sigma p_i x_i = 1$
$p_i x_i^2$	0	$\frac{1}{2}$	1	$\Sigma p_i x_i^2 = \frac{3}{2}$

$$\therefore \quad \text{Mean } \mu = \Sigma p_i x_i = 1$$

Variance $$\sigma^2 = \Sigma p_i x_i^2 - \mu^2 = \frac{3}{2} - (1)^2 = \frac{1}{2}$$

Standard Deviation $\sigma = \frac{1}{\sqrt{2}}$.

Q. 2. An insurance company insured 2,000 scooter drivers, 4,000 car drivers and 6,000 truck drivers. The probabilities of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured person meets with an accident. What is the probability that he is a scooter driver ? *(CBSE Delhi, 2000, 02, 03, 08)*

Solution :

Let we represent the set of scooter drivers be S, car drivers be C and truck drivers be T,

Then $$P(S) = \frac{2000}{2000+4000+6000} = \frac{2000}{12000} = \frac{1}{6}$$

$$P(C) = \frac{4000}{2000+4000+6000} = \frac{1}{3}$$

and $$P(T) = \frac{6,000}{2,000+4,000+6,000} = \frac{1}{2}$$

Let E = Event that accident occurs

$$P\left(\frac{E}{S}\right) = 0.01 = \frac{1}{100}, P\left(\frac{E}{C}\right) = 0.03 = \frac{3}{100}, P\left(\frac{E}{T}\right) = 0.15 = \frac{15}{100}$$

By Bays' Theorem,

$$P\left(\frac{E}{S}\right) = \frac{P(S)P\left(\frac{E}{S}\right)}{P(S)P\left(\frac{E}{S}\right) + P(C)P\left(\frac{E}{C}\right) + P(T).P\left(\frac{E}{T}\right)}$$

$$= \frac{\frac{1}{6}\left(\frac{1}{100}\right)}{\left(\frac{1}{6}\right)\left(\frac{1}{100}\right)+\left(\frac{1}{3}\right)\left(\frac{3}{100}\right)+\frac{1}{2}\left(\frac{15}{100}\right)}$$

$$= \frac{\frac{1}{600}}{\frac{1}{600}+\frac{3}{300}+\frac{15}{200}}$$

$$= \frac{\frac{1}{600}}{\frac{1+6+45}{600}} = \frac{\frac{1}{600}}{\frac{52}{600}} = \frac{1}{52}$$

Q. 3. A random variable X has the following probability function :

$X = x_i$	0	1	2	3	4	5	6	7
$P(x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

then find P(X < 6)

Solution :

We know that : $\Sigma P(X) = 1$

$\therefore P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$

$\Rightarrow \quad 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad 10k^2 + 10k - k - 1 = 0$

$\Rightarrow \quad 10k\,(k+1) - 1\,(k+1) = 0$

$\Rightarrow \quad (10k - 1)\,(k+1) = 0$

$$k = \frac{1}{10}, -1$$

Hence probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

$$P(X < 6) = P(0) + P(1) + P(2) + P(3) + P(4) + P(5)$$

$$= 0 + \frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}$$

$$= \frac{81}{100} = 0.81$$

Q. 4. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into the urn. Otherwise it is replaced into the urn along with another ball of the same colour. The process is repeated. Find the probability that the third ball drawn is black.

Solution :

Let we represent white ball be X and back ball be Y.

Then required probability = P (Third ball is black)

= P(XXY, XYY, YXY, YYY)

$$= \frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}+\frac{2}{4}\times\frac{2}{3}\times\frac{3}{4}+\frac{2}{4}\times\frac{2}{5}\times\frac{3}{4}+\frac{2}{5}\times\frac{3}{5}\times\frac{4}{6}$$

$$= \frac{1}{6}+\frac{1}{4}+\frac{3}{20}+\frac{1}{5}$$

$$= \frac{10+15+9+12}{60}$$

$$= \frac{46}{60} = \frac{23}{30}$$

Q. 5. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn at random and are found to both clubs. Find the probability of the lost card being of clubs.

(CBSE, Delhi, 2010)

Solution :

Let events of lost card being of clubs be E_1, E_2, E_3 and E_4, then

$$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$$

Let event A be drawing 2 club cards from ramaining 51 cards.

then $\quad P\left(\frac{A}{E_1}\right) = \frac{^{13}C_2}{^{51}C_2} = \frac{26}{425}$

$$P\left(\frac{A}{E_2}\right) = \frac{^{12}C_2}{^{51}C_2} = \frac{22}{425}$$

$$P\left(\frac{A}{E_3}\right) = \frac{^{13}C_2}{^{51}C_2} = \frac{26}{425}$$

and $\quad P\left(\frac{A}{E_4}\right) = \frac{^{13}C_2}{^{51}C_2} = \frac{26}{425}$

Now $\quad P\left(\frac{E_2}{A}\right) = \frac{P(E_2).P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)+P(E_3)P\left(\frac{A}{E_3}\right)+P(E_4)P\left(\frac{A}{E_4}\right)}$

$$= \frac{\frac{1}{4}\times\frac{22}{425}}{\frac{1}{4}\times\frac{26}{425}+\frac{1}{4}\times\frac{22}{425}+\frac{1}{4}\times\frac{26}{425}+\frac{1}{4}\times\frac{26}{425}}$$

$$= \frac{11}{13+11+13+13} = \frac{11}{50}$$

Q. 6. There are two bags, bag I and bag II. Bag I contains 4 white and 3 red balls while another bag II contains 3 white and 7 red balls. One ball is drawn at random from one of the bags and it is found to be white. Find the probability that it was drawn from bag I.

(CBSE, Delhi, 2010)

Solution :

Let E_1 : Drawn a ball from bag I

and E_2 : Drawn a ball from bag II

$$\therefore \quad P(E_1) = P(E_2) = \frac{1}{2}$$

Let E : Drawn a white ball

Then $P\left(\frac{E}{E_1}\right) = \frac{4}{7}$

and $P\left(\frac{E}{E_2}\right) = \frac{3}{10}$

Required probability,

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right)P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1)+P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{\frac{4}{7}\times\frac{1}{2}}{\frac{4}{7}\times\frac{1}{2}+\frac{3}{10}\times\frac{1}{2}}$$

$$= \frac{\frac{2}{7}}{\frac{2}{7}+\frac{3}{20}} = \frac{40}{61}$$

Q. 7. From a lot of 10 bulbs, which includes 3 defectives, a sample of 2 bulbs is drawn at random. Find the probability distribution of the number of defective bulbs. *(CBSE, Delhi, 2010)*

Solution :

Let X be a random variable and no. of defective bulbs be X.

$\therefore$ $X = 0, 1, 2, 3, ...$

When $X = 1$

i.e., 1 defective bulb and second bulb not defective.

$$P(X = 1) = 2\left(\frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2}\right)$$

$$= 2\left(\frac{2\times7\times3}{10\times9}\right) = \frac{14}{15}$$

When X = 2 *i.e.*, 2 bulbs is defective

$$\therefore \quad P(X = 2) = \frac{{}^3C_2}{{}^{10}C_2} = \frac{1}{15}$$

Probability distribution :

X	1	2
P(X)	$\frac{14}{15}$	$\frac{1}{15}$

Q. 8. Find the probability distribution of doublets in three throws of a pair dice. *(CBSE, 2005)*

Solution :

Let X represents the no. of doublets,

[(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)]

It is clear that the value of X takes 0, 1, 2 and 3.

If event A getting doublets, then $P(A) = \frac{6}{36} = \frac{1}{6}$

Not getting doublets $P(\overline{A}) = 1 - \frac{1}{6} = \frac{5}{6}$

$\therefore$ $P(X = 0) = P$ (no doublets)

$$= P(\overline{A}\overline{A}\overline{A}) = P(\overline{A}).P(\overline{A}).P(\overline{A})$$

$$= \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6} = \frac{125}{216}$$

$P(X = 1) = P$ (only doublets)

$$= \frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$$

$$= \frac{25}{216}+\frac{25}{216}+\frac{25}{216} = \frac{75}{216}$$

$P(X = 2) = P$ (two doublets)

$$= \frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6} = \frac{15}{216}$$

$P(X = 3) = P$ (three doublets)

$$= \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6} = \frac{1}{216}$$

Again $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= \frac{125}{216}+\frac{75}{216}+\frac{15}{216}+\frac{1}{216} = \frac{216}{216} = 1$$

Hence probability distribution of a random variable :

X	0	1	2	3
P(X)	125/216	75/216	15/216	1/216

Q. 9. There are three urns. Urn I contains 1 white, 2 black and 3 red balls. Urn II contains 2 white, 1 black and 1 red ball. Urn III contains 4 white, 5 black and 3 red balls. One urn is chosen at random and two balls drawn without replacement. They happen to be white and red. What is the probability that they are from urn I, II and III? *(CBSE, Delhi, 2009)*

Solution :

Let events of urns I, II and III respectively be E_1, E_2 and E_3.

$$\therefore \quad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let A = a white and a red ball

$$P(A/E_1) = \frac{1}{5},\ P(A/E_2) = \frac{1}{3},\ P(A/E_3) = \frac{2}{11}$$

Thus

$P(A) = P(E_1).P(A/E_1) + P(E_2).P(A/E_2) + P(E_3).P(A/E_3)$

$$= \frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{11}$$

$$= \frac{1}{15}+\frac{1}{9}+\frac{2}{33} = \frac{33+55+30}{495} = \frac{118}{495}$$

Thus

By Bays' Theorem required probability,

$$P(E_1A) = \frac{P(E_1).P(A/E_1)}{P(A)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{5}}{118/495} = \frac{1}{15}\times\frac{495}{118} = \frac{33}{118}$$

required probabily

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(A)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{3}}{118/495} = \frac{1}{9}\times\frac{495}{118} = \frac{55}{118}$$

and required probability

$$P(E_3/A) = \frac{P(E_3).P(A/E_3)}{P(A)}$$

$$= \frac{\frac{1}{3}\times\frac{2}{11}}{118/495}$$

$$= \frac{2}{33}\times\frac{495}{118} = \frac{30}{118}$$

Q. 10. Chances of A, B and C being selected as a manager of a firm are in the ratio 4 : 1 : 2 respectively. Their respective probabilities to introduce a radical change in marketing strategy are 0.3, 0.8 and 0.5. If the change does takes place, find the probability that it is due to the appointment of B or C. (*CBSE*, 2005)

Solution :

Let the probabilities as a manager of firm A, B and C be E_1, E_2 and E_3, then

$$P(E_1) = \frac{4}{4+1+2} = \frac{4}{7},\ P(E_2) = \frac{1}{4+1+2} = \frac{1}{7}$$

and $$P(E_3) = \frac{2}{4+1+2} = \frac{2}{7}$$

Let the event E for changing, then

$$P\left(\frac{E}{E_1}\right) = 0.3,\ P\left(\frac{E}{E_2}\right) = 0.8,\ P\left(\frac{E}{E_3}\right) = 0.5$$

By Bays' Theorem $P\left(\frac{E}{E_2}\right)+P\left(\frac{E}{E_3}\right)$

$$= \frac{P(E_2)P\left(\frac{E}{E_2}\right)+P(E_3)P\left(\frac{E}{E_3}\right)}{P(E_1)P\left(\frac{E}{E_1}\right)+P(E_2)P\left(\frac{E}{E_2}\right)+P(E_3)P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\left(\frac{1}{7}\right)(0.8)+\left(\frac{2}{7}\right)(0.5)}{\left(\frac{4}{7}\right)(0.3)+\left(\frac{1}{7}\right)(0.8)+\left(\frac{2}{7}\right)(0.5)}$$

$$= \frac{\left(\frac{8}{70}\right)+\left(\frac{2}{7}\right)\left(\frac{5}{10}\right)}{\left(\frac{4}{7}\right)\left(\frac{3}{10}\right)+\left(\frac{1}{7}\right)\left(\frac{8}{10}\right)+\left(\frac{2}{7}\right)\left(\frac{5}{10}\right)}$$

$$= \frac{\frac{18}{70}}{\frac{12+8+10}{70}} = \frac{18}{30} = \frac{3}{5} = 0.6$$

Q. 11. A pair of dice is thrown 4 times if getting a doublet is considered a success, find the probability of two success. (*CBSE, Delhi*, 2008)

Solution :

$$P = \frac{6}{36} = \frac{1}{6}$$

Then $$q = 1-\frac{1}{6} = \frac{5}{6}$$

$$P(X) = 2 = {}^4C_2\,(P)^{4-2}\left(\frac{5}{6}\right)^2$$

$$= \frac{\underline{|4}}{\underline{|2}\,\underline{|2}}\times\left(\frac{1}{6}\right)^2\times\frac{25}{36}$$

$$= 6\times\frac{1}{36}\times\frac{25}{36} = \frac{25}{216}.$$

Q. 12. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and noted the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability she threw 1, 2, 3 or 4 with the die ? (*CBSE, Outside Delhi*, 2012)

Solution :

If a die throws, then option 6 cases.

Probability getting 5 or 6

$$P(E_1) = \frac{2}{6} = \frac{1}{3}$$

i.e. $$P(E_1) = \frac{1}{3}$$

Probability getting 1, 2, 3, 4

$$P(E_2) = \frac{4}{6} = \frac{2}{3}$$

i.e. $$P(E_2) = \frac{2}{3}$$

If a coin tosses three times, then getting 5 or 6.

∴ [HHH, HHT, HTH, THH, HTT, THT, TTH, TTT]

Getting a head, *i.e.*, [HTT, THT, TTH]

Probability of getting a head, $P\left(\frac{A}{E_1}\right) = \frac{3}{8}$

Getting 1, 2, 3, 4 when a coin tosses once.

Probability of getting a head = $\frac{1}{2}$

i.e. $P(A/E_2) = \frac{1}{2}$

Probability of getting 1, 2, 3 or 4 throw by its, if a coin tosses obtain exactly one head = $P(E_2/A)$

$$\therefore\quad P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}} = \frac{1}{3}\div\frac{11}{24}$$

$$= \frac{1}{3}\times\frac{24}{11} = \frac{8}{11}.$$

Q. 13. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholar attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is hosteler ? *(CBSE, Delhi, 2012)*

Solution :

Let events of reside in hostel and day scholars students be E_1 and E_2.

Probability of students who residing in hostel = 60% = 0.6

Probability of students who is not residing in hostel = 40% = 0.4

Probability of students who residing in hostel attain A grade = 30% = 0.3 = P (A/E_1)

Probability of students who is not residing in hostel attain A grade = P (A/E_2) = 20% = 0.2

Probability of students who is residing in hostel attain A grade = P (E_1/A)

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{0.6\times0.3}{0.6\times0.3+0.4\times0.2}$$

$$= \frac{0.18}{0.18+0.08} = \frac{0.18}{0.26} = \frac{18}{26} = \frac{9}{13}.$$

Q. 14. A speaks the truth in 75% cases and B in 80% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact ? *(BSEB, 2015)*

Solution :

Since A is talling truth = 75%

and B is talling truth = 80%

thus $P(A) = \frac{75}{100} = \frac{3}{4}$

and $P(B) = \frac{80}{100} = \frac{4}{5}$

Hence, required probability $= 1 - P(\bar{A})P(\bar{B})$

$$= 1-\left(1-\frac{3}{4}\right)\cdot\left(1-\frac{4}{5}\right)$$

$$= 1-\frac{1}{4}\times\frac{1}{5}$$

$$= \frac{19}{20}$$

thus, probability $= \frac{19}{20}\times100 \times 100 = 95\%$

Q. 15. From a pack of 52 cards, two cards are drawn randomly one by one without replacement. Find the probability that both of them are red. *(Raj. Board, 2015)*

Solution :

Ways of drawn 2 cards from 52 cards = ${}^{52}C_2$

And there is 26 cards red in 52 cards, then probability of two cards are red = ${}^{26}C_2$

thus, Hence required probability $= \frac{{}^{26}C_2}{{}^{52}C_2}$

$$= \frac{\frac{26!}{2!24!}}{\frac{52!}{2!\,50!}} = \frac{\frac{26\times25}{2}}{\frac{52\times51}{2}} = \frac{13\times25}{26\times51}$$

$$= \frac{25}{102}$$

Q. 16. If a fair coin is tossed 10 times, find the probability of appearing exactly four tails. *(Raj. Board, 2015)*

Solution :

On trossing a coin 10 times, let number of appearing tails and frequency, $n = 10$ and $p = \frac{1}{2}$

$\therefore$ $P(X = x) = {}^{n}C_x\, q^{n-x} . p^x$

Here $n = 10, p = \frac{1}{2}, q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

$\therefore$ $P(X = x) = {}^{10}C_x\left(\frac{1}{2}\right)^{10-x}\left(\frac{1}{2}\right)^{x}$

$$= {}^{10}C_x\left(\frac{1}{2}\right)^{10}$$

Since P (exactly four tails)

$$= P(X = 4)$$

$$= {}^{10}C_4\left(\frac{1}{2}\right)^{10}$$

$$= \frac{10!}{4!6!}\cdot\frac{1}{2^{10}}$$

$$= \frac{10^5}{256}$$

Q. 17. Let A and B be two events such that 2P(A) = P(B) = $\frac{5}{13}$ and P(A/B) = $\frac{2}{5}$. Find P(A $\cup$ B). *(JAC, 2015)*

Solution :

$\because$ $2P(A) = P(B) = \frac{5}{13}$

$\therefore$ $P(A) = \frac{5}{26}$

and $\quad P(A/B) = \frac{2}{5}$

$\therefore \quad P(A \cap B) = P(A/B) . P(B)$

$$= \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$$

$$= \frac{5 + 10 - 4}{26} = \frac{11}{26}$$

NCERT QUESTIONS

Q. 1. Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6.

Find : (i) P(A and B) (ii) P(A and B not)

(iii) P(A or B) (iv) P(Neither A or B)

Solution :

$P(A) = 0.3$

$P(B) = 0.6$

(i) $\quad P(A \cap B) = P(A).P(B)$

[∵ A and B are independents}

$P(A \cap B) = 0.3 \times 0.6 = 0.18$

(ii) P(A and B not) $= P(A \cap \overline{B})$

$= P(A) - P(A \cap B) = 0.3 - 0.18 = 0.12.$

(iii) $\quad$ P (A or B) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.6 - 0.18 = 0.9 - 0.18 = 0.72$

(iv) P (neither A nor B)

$= P(\overline{A} \cap \overline{B}) = P(A \cup B)$

$= 1 - P(A \cup B) = 1 - 0.72 = 0.28.$

Q. 2. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bag is selected and a ball is drawn at random from the bag which is found to be red. Find the probability that it is drawn from first bag.

Solution :

Let, E_1 : Probability of selected bag I

E_2 : Probability of selected bag II

then E_1 and E_2 are independent events

$P(E_1) = P(E_2) = \frac{1}{2}.$

Let E : drawn red ball

∴ Required probability

$$P(E_1/E) = \frac{P(E/E_1)P(E_1)}{P(E_1)P(E/E_1) + P(E_2)P(E/E_2)}$$

$$= \frac{\frac{4}{8} \times \frac{1}{2}}{\frac{4}{8} \times \frac{1}{2} + \frac{2}{8} \times \frac{1}{2}} = \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3}.$$

Probability drawn red ball from bag II $= \frac{1}{2} \times \frac{2}{8} = \frac{1}{8}$

Let event E drawn a white ball, then

$$P(E) = \frac{P(A)P(A/E)}{P(A)P(A/E) + P(B)P(B/E)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4}} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{8}} = \frac{2}{3}.$$

Q. 3. Given that A and B are events such that $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$ and P(B) = *p*, find *p* if they are (i) mutually exclusive, (ii) independent.

Solution :

(i) If A and B are mutually exclusive

$(A \cap B) = \phi$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$$0 = \frac{1}{2} + p - \frac{3}{5}$$

$$\Rightarrow \quad p = \frac{3}{5} - \frac{1}{2} = \frac{6-5}{10} = \frac{1}{10}$$

(ii) For independent,

$P(A \cap B) = P(A). P(B)$

but $\quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\Rightarrow \quad P(A) P(B) = P(A) + P(B) - P(A \cup B)$

$$\Rightarrow \quad \frac{1}{2} \times p = \frac{1}{2} + p - \frac{3}{5}$$

$$\Rightarrow \quad \frac{p}{2} = \frac{1}{2} + p - \frac{3}{5}$$

$$\Rightarrow \quad p - \frac{p}{2} = \frac{3}{5} - \frac{1}{2}$$

$$\Rightarrow \quad \frac{p}{2} = \frac{1}{10}$$

$$\therefore \quad p = \frac{1}{5}.$$

Q. 4. The probability that a bulb produced by a factory will fuse after 150 days of use is 0.05, find the probability that out of 5 such bulbs

(i) none

(ii) not more than one

(iii) more than one

(iv) atleast one will fuse after 150 days of use.

Solution

$$P = \frac{5}{100} = \frac{1}{20}$$

$$\Rightarrow \quad q = 1 - \frac{1}{20} = \frac{19}{20} = 0.95$$

Now (i) $\quad P(X = 0) = {}^5C_0(0.95)^5 = (0.95)^5$

(ii) $\quad$ P (not more than one) = PX = 0 and X = 1

$= (0.95)^4 [0.95 + 5 \times 0.05]$

$= (0.95)^4 (1.2)$

(iii) $\quad$ More than one $= [(1 - P(X = 1)]$

$= 1 - (0.95)^4 (1.2)$

(iv) $\quad$ At least one will fuse $= \{1 - P(X = 0)\}$

$= 1 - {}^5C_0 (0.95)^5$

$= 1 - (0.95)^2.$

■■